ACHARYA NARENDRA DEV COLLEGE

ELITE PROJECT - EXTENSION IN QUATERNION GROUP

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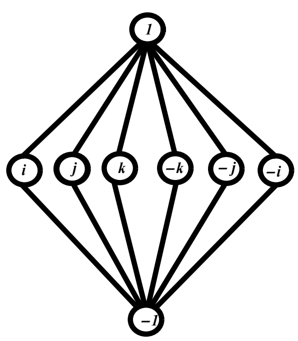
ABSTRACT

In this project, we have worked on Quaternions. We created the quaternion field and space. We have integrated quaternions with some examples of nature. Rotation of the earth is a quaternion ring, cyclooctane constructed by CH_2 and CH are similar to the quaternion octane by taking $\pm 1 = \pm i = \pm j = \pm k = CH_2$ and CH. The movement of a Cubesat is a group action on the quaternions and a cubesat itself is a quaternion space. Even the movement of a football is a rotation of the quaternions.

INTRODUCTION

Quaternion group consist of 8 elements namely **±1 , ±i , ±j , ±k,** where $i^2 = j^2 = k^2 = -1$, ij = k, jk = i , ki = j , ji = -k , kj = -i , ik = -j .



Q_8 is a non commutative group as ij ≠ ji

PROPERTIES

1. Q_8 has 6 subgroups.
2. Q_8 is non-cyclic but every proper subgroup of Q_8 is cyclic.
3. Every proper subgroup of Q_8 is normal.
4. Every proper subgroup of Q_8 is abelian.
5. $Z(Q_8) = \{1,-1\}$.

QUATERNION RING

Definition:- A non-empty set $Q_8=\{a+bi+cj+dk \mid a,b,c,d \in R, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j\}$,where R is any ring, is the ring as
The set Q_8
1. is an abelian group under addition.
2. is a semi group under multiplication.
3. satisfies left and right distributive properties.

And this ring is known as quaternion ring.

QUATERNION FIELD

A non-empty set $Q_8=\{a+bi+cj+dk \mid a,b,c,d \in F, i^2 = j^2=k^2= -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$, where F is a field$\}$, is a skew field as
the set Q_8
1. is an abelian group under addition.
2. is a semi group under multiplication.
3. satisfies left and right distributive properties.

and
We have the multiplicative identity of $(Q_8,\cdot)$ is $1 = 1 + 0\,(i+j+k)$
with the inverse of $a = 1/a \ \forall\ a \neq 0$.
and inverse of $a + bi = (a - bi)/(a^2+b^2)$ where either a or b is nonzero.
and inverse of $a +bi +cj = (a-bi-cj)/(a^2+b^2+c^2)$ where either a,b or c is nonzero.
and inverse of $a+bi+cj+dk = (a-bi-cj-dk)/(a^2+b^2+c^2+d^2)$ where either a,b,c or d is nonzero.

Since every element of Q_8 has its inverse in Q_8 , Q_8 is a division ring as Q_8 is non-abelian.
Therefore,
Q_8 is a ***skew field****. This is to be called Quaternion field.*

QUATERNION SPACE

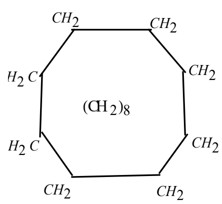
Vector Space:-Let S be a non-empty set , then the set S with two binary operations vector addition and scalar multiplication (.), is said to be a vector space over a field F(=R) if the following properties hold for every $A,B,C\in S$ and $a,b\in R$:-
1.(S,+) forms an abelian group.
2.S is closed under scalar multiplication.
3.S satisfies the associative property with scalar multiplication.
4. For $\forall\ x\in S$, $1.x=x$.
6. S satisfies the distributive property of scalar multiplication over vector addition.
7.S satisfies the distributive property of vector multiplication over scalar addition.

$Q_8=\{a+bi+cj+dk \mid a,b,c,d \in F, i^2 = j^2=k^2= -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$, where F is a field$\}$ satisfies the above mentioned properties, therefore it forms a vector space over F. This is to be called quaternion space.

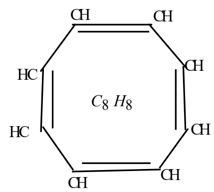
APPLICATIONS

APPROACH IN CHEMISTRY

Conformation of organic cyclic compounds : - *In organic compounds such as cyclooctatetraene* (C_8 H_8) and cyclooctane $(CH_2)_8$. *We can relate with a closed figure with 8 vertices where every Carbon can be compared with a distinct element of* Q_8.

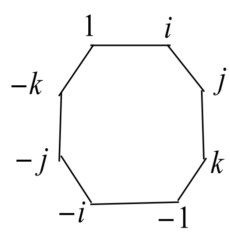


Cyclooctane



Cyclooctatetraene

Both figures can be compared with this

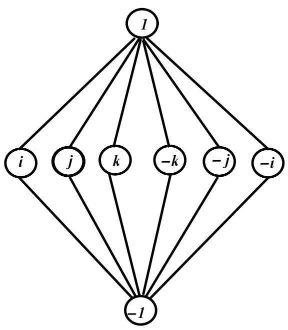


Where $|\pm 1| = |\pm i| = |\pm j| = |\pm k|$ = **one mole of CH_2 in first case and one mole of CH in second case, and $\pm 1 = \pm i = \pm j = \pm k = CH_2$ in first case and CH in second case .**

APPROACH IN GEOGRAPHY

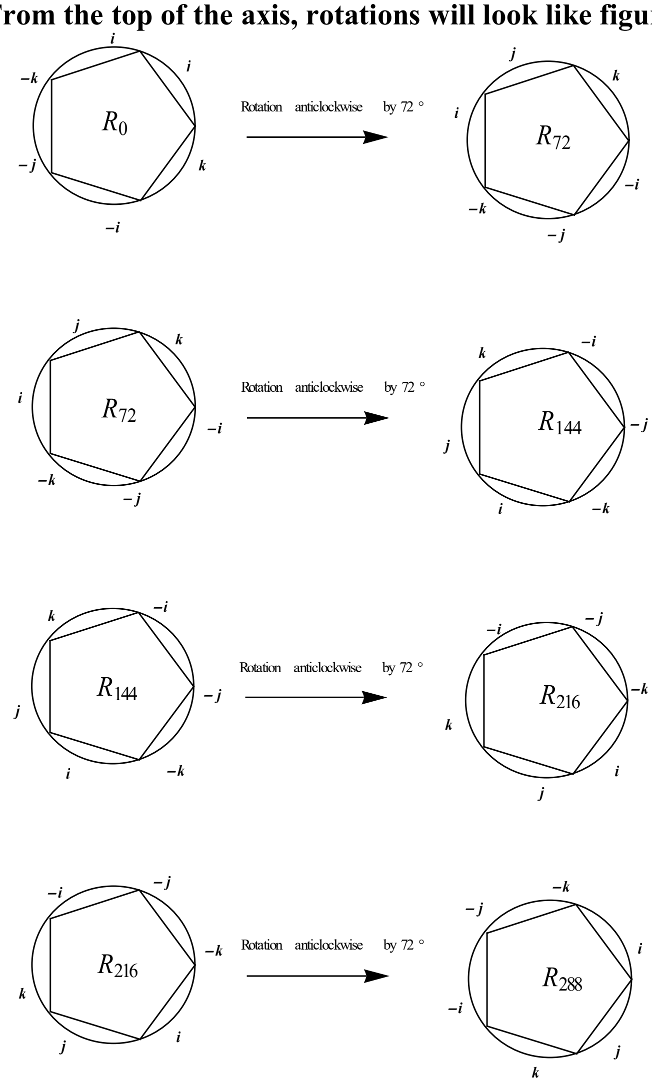
Globe:- A globe is a 3-D spherical, scale model of earth or other celestial body such as a planet or moon. We know that globe is a model of earth, where line drawn by joining north pole and south pole plays the role of axis of rotation.

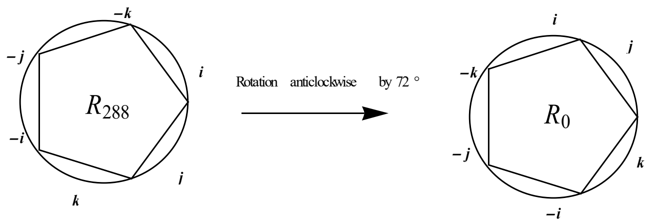
We can arrange the quaternions in the form given below



This form is similar to the globe, though globe is bent with an angle of 231/2°,where ±1 stands for two poles and ±i, ±j, ±k are the points at equator line. If we rotate earth in step of 72°, then every point will at the place as it was initially, after 5th rotation as $x^5=x$, $\forall$ $x \in Q_8$

From the top of the axis, rotations will look like figures given below:-





GROUP ACTION

Definition:- A group action of a group G on a set A is a map from G×A to A satisfying the following properties:

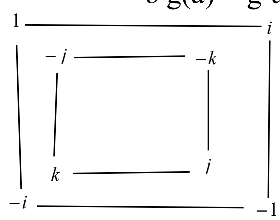
(1) $g_1\cdot(g_2\cdot a)=(g_1\ g_2)\cdot a$ for all $g_1,g_2\in G, a\in A$, and

(2) $1\cdot a = a$, for all $a \in A$

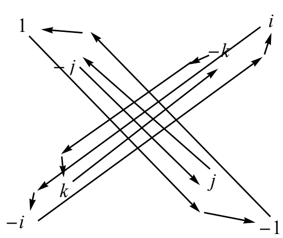
Let the group G acts on a set A. For each fixed g ϵ G, map σg defined by

$\sigma g : A \rightarrow A$

$\sigma g(a) = g\cdot a$

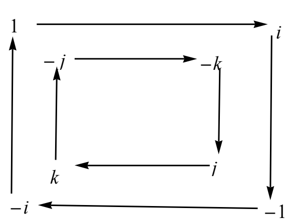


$\sigma 1(a)=1\cdot a= a$, where $a\in Q_8$



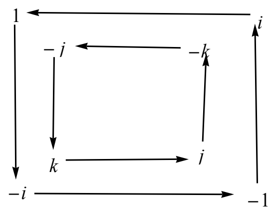
$(1\ -1)(i\ -i)(j\ -j)(k\ -k)$

$\sigma\text{-}1(a)=-1\cdot a$, where $a\in Q_8$



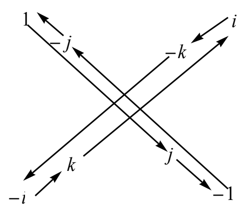
$(1\ i\ -1\ -i)(j\ k\ -j\ -k)$

$\sigma i(a)=i\cdot a$, where $a\in Q_8$



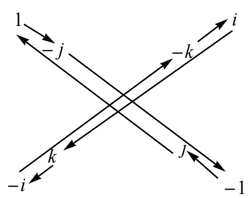
$(1\ \ -i\ \ -1\ \ i)(j\ \ -k\ \ -j\ \ k)$

$\sigma_{-i}(a)=-i\cdot a$, where $a\in Q_8$



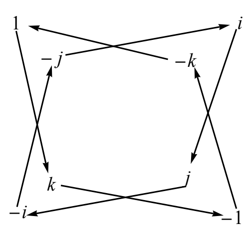
$(1\ \ j\ \ -1\ \ -j)(i\ \ -k\ \ -i\ \ k)$

$\sigma_j(a)=j\cdot a$, where $a\in Q_8$



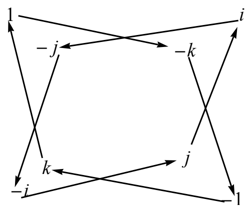
$(1\ \ -j\ \ -1\ \ j)(i\ \ k\ \ -i\ \ -k)$

$\sigma_{-j}(a)=-j\cdot a$, where $a\in Q_8$



$(1\ \ k\ \ -1\ \ -k)(i\ \ j\ \ -i\ \ -j)$

$\sigma_k(a)=k\cdot a$, where $a\in Q_8$



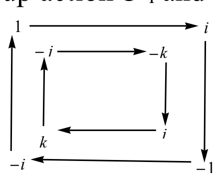
$(1\ \ -k\ \ -1\ \ k)(i\ \ -j\ \ -i\ \ j)$

$$\sigma_{-k}(a) = -k \cdot a \text{ , where } a \in Q_8$$

APPROACH IN PHYSICS

Cubesat :- A cubesat is a type of small satellite for space research that is made up of multiple 10×10×10 cm cubic units. Cubesats have a mass no more than 1.33 Kg per unit. Cubesats are most commonly put in orbit by deployers on the International Space Station. Basically, it has 8 vertices that's why it can be compared to Q_8.

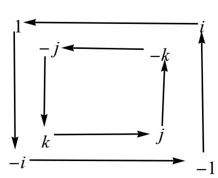
The movement of the above satellite can be studied by the group action of $\mathbf{Q_8}$ on itself, This can be understood by labelling 8 vertices of cubesat as ±1, ±i, ±j, ±k same as the above diagram of σi and σ-i . Cubesat will rotate in clockwise direction and anticlockwise through group action $\boldsymbol{\sigma}_{i}$ and $\boldsymbol{\sigma}_{-i}$.



$(1\ \ i\ \ -1\ \ -i)(j\ \ k\ \ -j\ \ -k)$

Clockwise Rotation

$\sigma_i(x)$

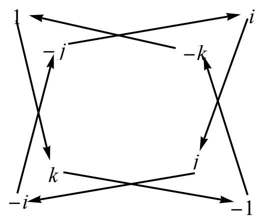


$(1\ \ -i\ \ -1\ \ i)(j\ \ -k\ \ -j\ \ k)$

Anticlockwise Rotation

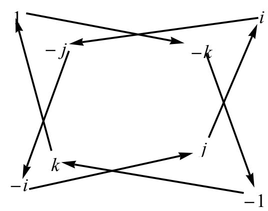
$\sigma_{-i}(x)$

Application of rotation of Q_8 in sports:- The movement of football can be compared with group action of Q_8 on itself, This can be understood by the rotations due to group actions $\boldsymbol{\sigma}_k$ and $\boldsymbol{\sigma}_{-k}$.



$(1\ k\ -1\ -k)(i\ j\ -i\ -j)$

$\sigma_k(x)$



$(1\ -k\ -1\ k)(i\ -j\ -i\ j)$

$\sigma_{-k}(x)$

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CONCLUSION

We have extended Quaternion ring to quaternion field and to quaternion space. Apart from this, we went through some of its applications, which behaved as some of the applications of Physics, Chemistry, and Geography .

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