

Gram -Schmidt Process
Dr. Sarita Agarwal
Assistant Professor
Department of Mathematics
Acharya Narendra Dev College
(University of Delhi)
Mail-saritaagarwal@andc.du.ac.in

Theorem: Let V be an inner product space and $S = \{w_1, w_2, \dots, w_n\}$ be a linearly independent subset of V . Define $S' = \{v_1, v_2, \dots, v_n\}$ where $v_1 = w_1$ and $v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$ for $2 \leq k \leq n$. Then S' is an orthogonal set of non-zero vectors such that $\text{Span}(S') = \text{Span}(S)$.

Proof: We prove the statement by induction on n .

For $n = 1$, $S'_1 = S_1 \Rightarrow v_1 = w_1$, and $w_1 \neq 0$ as $S_1 = \{w_1\}$ is linearly independent.

Assume that $S_{k-1}' = \{v_1, v_2, \dots, v_{k-1}\}$ has been constructed with the desired properties.

Claim: $S_k' = \{v_1, v_2, \dots, v_k\}$ is orthogonal with the desired properties while $S_k = \{w_1, w_2, \dots, w_k\}$ is a linearly independent subset of V .

Since $S_{k-1}' = \{v_1, v_2, \dots, v_{k-1}\}$ has been constructed with the desired properties therefore $v_1 \neq 0, v_2 \neq 0, \dots, v_{k-1} \neq 0$ and if $v_k = 0$, then

$$w_k = \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, \text{ contradicts the fact that } S_k \text{ is linearly independent set.}$$

$\therefore v_k \neq 0$.

$$\text{Consider, } \langle v_k, v_i \rangle = \langle w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, v_i \rangle$$

$$= \langle w_k, v_i \rangle - \langle \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, v_i \rangle$$

$$= \langle w_k, v_i \rangle - \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} \|v_j\|^2 \text{ (because } S_{k-1}' = \{v_1, v_2, \dots, v_{k-1}\} \text{ is an orthogonal set.)}$$

set.)

$$= 0.$$

$$\Rightarrow S_k' \text{ is an orthogonal subset of } V \text{ with the property } v_1 = w_1 \text{ and } v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n.$$

$$\Rightarrow \text{Span}(S_k') \subseteq \text{Span}(S_k) \text{ and } \dim(\text{Span}(S_k')) = k = \dim(\text{Span}(S_k)).$$

Thus $\text{Span}(S_k') = \text{Span}(S_k)$. Hence according to induction hypothesis, the statement is true for $n = k$.

Remark: The process of converting the linearly independent set into orthogonal is called Gram-Schmidt Process.

Theorem: Let V be a non-zero finite dimensional inner product space. Then V has an orthonormal basis β . Furthermore if $\beta = \{v_1, v_2, \dots, v_n\}$ and $x \in V$, then $x = \sum_{j=1}^n \langle x, v_j \rangle v_j$.

Proof: Let α be the basis of V and thus is a linearly independent subset of V . Gram Schmidt process converts into orthogonal set γ such that $V = \text{Span}(\alpha) = \text{Span}(\gamma)$. So, γ is orthogonal basis for V . Normalize the basis set γ to get the orthonormal basis β for V .

Clearly, if $x \in V$, then $x = \sum_{j=1}^n \langle x, v_j \rangle v_j$.

Remark: Let V be a finite dimensional inner product space with an orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$. Let T be a linear operator on V and let $A = [T]_\beta$. Then for any i and j , $A_{ij} = \langle T(v_j), v_i \rangle$. But for $T(v_j) \in V$, then $T(v_j) = \sum_{i=1}^n \langle T(v_j), v_i \rangle v_i$. Therefore $A_{ij} = \langle T(v_j), v_i \rangle$.

Example: Find the orthonormal basis for $P_2(\mathbb{R})$.

Vector space $P_2(\mathbb{R})$ with the product $\langle f(x), g(x) \rangle = \int_{-1}^1 f(t)g(t)dt$ is an inner product space over \mathbb{R} .

$\beta = \{1, x, x^2\}$ is the standard ordered basis for $P_2(\mathbb{R})$.

Gram Schmidt process gives us orthogonal basis $\gamma = \{v_1, v_2, v_3\}$

Where $v_1 = 1$, $\|v_1\|^2 = \langle v_1, v_1 \rangle = \int_{-1}^1 1^2 dt = 2$ and $\langle v_1, x \rangle = \int_{-1}^1 t \cdot 1 dt = 0$.

Thus $v_2 = x - \frac{\langle v_1, x \rangle}{\|v_1\|^2} v_1 = x$ and $\|v_2\|^2 = \langle v_2, v_2 \rangle = \int_{-1}^1 t^2 dt = \frac{2}{3}$

$\langle v_1, x^2 \rangle = \int_{-1}^1 1 \cdot t^2 dt = \frac{2}{3}$ and $\langle v_2, x^2 \rangle = \int_{-1}^1 t \cdot t^2 dt = 0$.

$v_3 = x^2 - \frac{\langle v_1, x^2 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_2, x^2 \rangle}{\|v_2\|^2} v_2 = x^2 - \frac{1}{3}$.

Normalize to get orthonormal:

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{3}{2}} x$$

$$u_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{5}{8}} (3x^2 - 8)$$

$\{u_1, u_2, u_3\}$ is the desired orthonormal basis for $P_2(\mathbb{R})$.

Remark:

(1). We can write a polynomial $f(x) = 1 + 2x + 3x^2$ as the linear combination of vectors in the orthonormal basis $\{u_1, u_2, u_3\}$ for $P_2(\mathbb{R})$.

$f(x) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$, where $\alpha_i = \langle f(x), u_i \rangle$ for $i=1, 2, 3$

$$\langle f(x), u_1 \rangle = \int_{-1}^1 \frac{1}{\sqrt{2}} (1 + 2t + 3t^2) dt = 2\sqrt{2}$$

$$\langle f(x), u_2 \rangle = \int_{-1}^1 \sqrt{\frac{3}{2}} t (1 + 2t + 3t^2) dt = \frac{2\sqrt{6}}{3}$$

$$\langle f(x), u_3 \rangle = \int_{-1}^1 \sqrt{\frac{5}{8}} (3t^2 - 1)(1 + 2t + 3t^2) dt = \frac{2\sqrt{10}}{5}.$$

$$f(x) = \langle f(x), u_1 \rangle u_1 + \langle f(x), u_2 \rangle u_2 + \langle f(x), u_3 \rangle u_3.$$

(2). Similarly, we can compute orthogonal projection of $f(x)=x^3$ on $P_2(\mathbb{R})$.

$$f_1(x) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \text{ where } \alpha_i = \langle f(x), u_i \rangle \text{ for } i=1,2,3.$$

$$\langle f(x), u_1 \rangle = \int_{-1}^1 \frac{1}{\sqrt{2}} t^3 dt = 0.$$

$$\langle f(x), u_2 \rangle = \int_{-1}^1 \sqrt{\frac{3}{2}} t t^3 dt = \frac{\sqrt{6}}{5}.$$

$$\langle f(x), u_3 \rangle = \int_{-1}^1 \sqrt{\frac{5}{8}} (3t^2 - 1)t^3 dt = 0.$$

$$f_1(x) = \langle f(x), u_1 \rangle u_1 + \langle f(x), u_2 \rangle u_2 + \langle f(x), u_3 \rangle u_3 = \frac{3}{5}x, \text{ orthogonal projection on } P_2(\mathbb{R}).$$

Definition: Let β be an orthonormal subset of an inner product space V , for $x \in V$, the Fourier coefficients of x relative to β is the scalar $\langle x, y \rangle$, $y \in \beta$.

Definition: Let S be a non-empty subset of an inner product space V , then the orthogonal complement of S is

$$S^\perp = \{ x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S \}.$$

Clearly S^\perp is a subspace of V .(?)

Remark:

(1). $\{0\}^\perp = V$ and $V^\perp = \{0\}$ for an inner product space V .

(2). If $V = \mathbb{R}^2$ and $S = \{e_2\}$ and $S^\perp = \{e_1\}$.

(3). If W is a finite-dimensional subspace of an inner product space V , then

$$(a) W = (W^\perp)^\perp.$$

$$\text{Let } u \in W, \text{ then } \langle u, z \rangle = 0, \forall z \in W^\perp.$$

$$\Leftrightarrow u \in (W^\perp)^\perp.$$

$$(b) W \cap W^\perp = \{0\}.$$

Theorem: Let W be a finite-dimensional subspace of an inner product space V with orthonormal basis $\{v_1, v_2, \dots, v_k\}$, then for $y \in V$, there exists unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$ if and only if $u = \sum_{j=1}^k \langle y, v_j \rangle v_j$.

Proof: Let W be a finite-dimensional subspace of an inner product space V with orthonormal basis $\{v_1, v_2, \dots, v_k\}$, then for $y \in V$, there exists unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$.

$$\text{Claim: } u = \sum_{j=1}^k \langle y, v_j \rangle v_j.$$

Since $y = u + z$

$$\therefore u = y - z$$

$$u = \sum_{j=1}^k \langle u, v_j \rangle v_j = \sum_{j=1}^k \langle y - z, v_j \rangle v_j$$

$$= \sum_{j=1}^k \langle y, v_j \rangle v_j - \sum_{j=1}^k \langle z, v_j \rangle v_j = \sum_{j=1}^k \langle y, v_j \rangle v_j (\because z \in W^\perp)$$

Conversely, let $u = \sum_{j=1}^k \langle y, v_j \rangle v_j$

Claim: There exists unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$.

Consider, $\langle z, v_i \rangle = \langle y - u, v_i \rangle = \langle y, v_i \rangle - \langle u, v_i \rangle = \langle y, v_i \rangle - \langle \sum_{j=1}^k \langle y, v_j \rangle v_j, v_i \rangle$

$$= \langle y, v_i \rangle - \langle y, v_i \rangle = 0$$

($\because \{v_1, v_2, \dots, v_k\}$ is orthonormal basis set for W).

$$\Rightarrow z \perp v_i$$

$v_i \in \{v_1, v_2, \dots, v_k\}$ is arbitrary and thus $z \perp \{v_1, v_2, \dots, v_k\}$, orthonormal basis set for W .

$$\Rightarrow z \perp x, \forall x \in W \text{ and thus } z \in W^\perp.$$

$$\langle u, z \rangle = \langle \sum_{j=1}^k \langle y, v_j \rangle v_j, z \rangle = \sum_{j=1}^k \langle y, v_j \rangle \langle v_j, z \rangle = 0 \Rightarrow u \in W.$$

Uniqueness

Let $y = u + z$ and $y = u' + z' \Rightarrow u - u' = z - z' \in W \cap W^\perp = \{0\}$.

$$\Rightarrow u = u' \text{ and } z = z'.$$

Theorem: If $S = \{v_1, v_2, \dots, v_k\}$ is an orthonormal set in a finite dimensional inner product space V , then

- (1) S can be extended to an orthonormal basis for V .
- (2) If $W = \text{Span}(S)$, then there exists a subset S' of orthonormal basis for W^\perp .
- (3) If W is a subspace of V , then $V = W \oplus W^\perp$.

Proof: Let $S = \{v_1, v_2, \dots, v_k\}$ is an orthonormal set in an n -dimensional inner product space V . S being an orthonormal subset of V is linearly independent.

- (1) According to the Replacement Theorem S can be extended to the basis set $\beta = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V . β being the basis set for V is linearly independent, is easily converted into orthogonal basis β' for V (using Gram-Schmidt Process). Normalize β' to get orthonormal basis β'' for V .
- (2) Let $W = \text{Span}(S)$. Then S is a basis set for W .
 W^\perp is a subspace of V .
Let $S' = \{v_{k+1}, \dots, v_n\}$
Let $z \in W^\perp \subseteq V$, we have $z = \sum_{j=1}^n \langle z, v_j \rangle v_j$.
 $z \in W^\perp \Rightarrow \langle z, v_j \rangle = 0 \quad \forall j=1, 2, \dots, k$
Thus $z = \sum_{j=k+1}^n \langle z, v_j \rangle v_j$ and $z \in W^\perp$ is arbitrary.

So $W^\perp = \text{Span}(S')$ and S' being the subset of linearly independent set β is linearly independent and thus the basis for W^\perp . Using the Gram- Schmidt process and normalizing, we get orthonormal basis for W^\perp .

- (3). Since for each y in V , there exists unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$, therefore if W is a subspace of V then $V = W + W^\perp$ and $W \cap W^\perp = \{0\}$ implies $V = W \oplus W^\perp$.

Remark: If $\{e_1, e_2\}$ is a basis set for W in F^3 , then $\{e_3\}$ is a basis set for W^\perp

Reference

Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, "Linear Algebra" Fourth edition Pearson Education, 2003.